Building Signals with Blocks:
Basis Expansions

AllSignalProcessing.com

Key Concepts

1) Complicated signals are often described as weighted sums (or integrals) of elementary signals or "bases".

2) The weights are an equivalent description for the signal assuming the bases are known.

3) Any signal with N values can be represented exactly using N linearly independent basis signals.

4) Signal processing is performed by modifying the weights associated with the basis decomposition.

5) Signal processing of basis expansions works particularly well when the signal of interest is associated with a small subset of weights.

a) Noise and interference are reduced by zeroing out all weights not associated with the signal.

b) The signal can be compressed by only storing the weights associated with the signal.

6) Good bases tend to represent the signal of interest with relatively small number of non-zero weights.

7) There is no universally optimal set of basis signals.

8) Examples of useful bases include sinusoids of different frequencies and wavelets, i.e., limited-duration oscillatory signals.

Transcript

Building Signals with Blocks: Basis Expansions

Introduction

This lesson introduces one of the most profound and widespread concepts in signal processing: building complex signals from elementary signals.

We have all seen or played with building bricks – blocks with very simple shapes that can be combined to build almost unimaginably complex objects. Model skyscrapers, space vehicles, robots, and ships are some of the objects that can be constructed; the possibilities are endless.

Similarly, an orchestra produces a complex musical piece by combining simpler sounds from violins, cellos, clarinets, trumpets, horns, and so on.

The same concepts are used in signal processing. Complex signals can be constructed from combinations of very basic or building block signals. We often start with a complicated signal and decompose it into a combination of elementary signals.

This decomposition is analogous to taking the musical piece from an orchestra and identifying the parts played by each instrument.

Decomposition of arbitrary signals into elementary signals is a key part of many signal-processing techniques, including filtering, compression, denoising, and equalization.

Weighted sums of signals

We will call our collection of elementary building block signals “bases” and use “basis” when referring to one of the signals in the collection. This terminology is intimately tied to the notion of bases in the field of linear algebra.

Arbitrary signals are represented as weighted sums of basis signals or bases. That is, we express an arbitrary signal by scaling each basis with a weight and then adding it to the scaled versions of all the other bases. The weights associated with each basis signal are another way of describing the signal.

It turns out that an arbitrary signal with N values can be exactly represented this way provided we use N linearly independent basis signals. The term linear independent has a specific mathematical definition, but for our purposes here it means that the bases are sufficiently different from each other. That is, there is no duplication in the set of basis signals.

Signal processing is performed by modifying the weights associated with each basis.

In some cases arbitrary signals are represented by a weighted integral of basis signals. An integral is nothing more than the limiting form of a sum, so the weighted sum viewpoint also applies to integral expansions.

The conversion of signal values into the equivalent set of weights is usually termed a "transform". Common signal processing examples include the discrete Fourier transform, the discrete wavelet transform and the discrete cosine tranform.

Which basis?

The best choice of basis signals for representing arbitrary signals is application dependent. *There is no universally optimal set of basis signals.* This lack of optimality is the reason there are so many different basis decomposition techniques.

One common class of bases, and often the first encountered in the study of signal processing, is sinusoids. Sinusoids of different frequencies are the basis signals used in Fourier transforms and analysis.

Joseph Fourier (1768-1830) was a French mathematician and physicist. He is best known for developing a technique for solving partial differential equations associated with heat flow and vibration. He proposed solutions involving a weighted sum of sinusoids of distinct frequencies. His work was controversial at the time; now his ideas are widely used in engineering and science.

Sinusoids of different frequencies are very easy to visualize and are ubiquitous in the natural world. For example, the electromagnetic spectrum consists of sinusoidal waves of different frequency. Notes in music have different frequencies.

Sinusoids also possess a very powerful property. Sinusoids are "eigenfunctions" of linear time invariant systems. Linear time invariant systems are widely used in signal processing.

The eigenfunction property means that if you put a sinusoid into a linear time invariant system, the output is also a sinusoid of the same frequency. The system only modifies the amplitude and phase of the input sinusoid. This property leads to an insightful and powerful view of filtering.

Another widely used class of basis signals involves wavelets. Wavelets are limited duration oscillatory signals. The "frequency" of the oscillation is inversely related to the duration – longer duration wavelets have lower frequency.

A collection of wavelets of characteristic shape but with different durations and onset locations forms a basis. Different wavelet bases are obtained by changing the shape of the oscillation. There are dozens of different types of wavelets.

Wavelet bases have many desirable properties. They naturally trade temporal resolution and frequency resolution. Long duration wavelets tend to accurately capture the frequency of low-frequency signal features. Short duration wavelets tend to accurately capture the location of transient events.

The limited duration of wavelets make them well suited to represent signals whose characteristics change over time or space.

There are many other types of bases – too many to catalog here.

In general, a good basis for a particular application, or class of signals, will concentrate the energy of the signal into a relatively small number of basis signals. That is, the weights associated with many of the basis signals are negligible – a relatively small number of bases have significant weights. Hence different types of bases are best for different types of signals.

In some cases the statistics of the data are used to find a good set of basis signals. Examples include principle component analysis and independent component analysis.

Using basis expansions

Many of the signal-processing techniques for using basis expansions rely on the ability of the basis to concentrate the energy of the signal into a relatively small number of basis signals.

For example, suppose we measure a mixture of two signals, but that each signal can be expressed using different subsets of the basis signals. In this case we can perfectly separate the two signals. We simply set all basis weights to zero except for the subset representing the signal of interest.

Similarly, suppose we measure a signal in noise, and that the signal of interest is concentrated in a known subset of the basis signals. Typically noise energy is distributed across all bases. The signal to noise ratio is improved by zeroing out the weights associated with bases having little or no signal of interest.

Signal compression also exploits the concentration idea. If many of the weights in the basis expansion for a signal are very small, then we can "compress" the signal by only storing the large weights and indices of the corresponding bases. If a signal has N values, but can be represented using $p << N$ bases, then we achieve a compression ratio of approximately N to p. JPEG compression of images is based on this idea. Compression ratios 10 to 1 are often obtained in JPEG with no visible distinction between original and compressed images.

Summary

Basis expansions for signals have a myriad of uses in signal processing. The general idea is to express an arbitrary signal as a weighted sum (or integral) of basis signals. There are many different types of bases. Good bases concentrate the signal information into a small number of weights.

The big picture view presented in this lesson will make it much easier to understand basis expansions and their role in signal processing.